AP Calculus BC

Unit 5 – Analytical Applications of Differentiation

AP Calculus BC – Worksheet 31 **Extreme Values of Functions**

1

Let *f* be the function defined on $[-1,2]$ by $f(x)=3x^{\frac{2}{3}}-2x$.

a) Find the domain of $f(x)$.

b) Find $f'(x)$ and the domain of $f'(x)$.

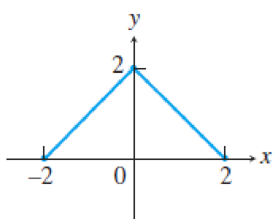
c) Determine if the EVT applies to $f(x)$ on the given interval. If it does, find the max and min values guaranteed by the theorem. Justify using the Closed Interval Argument. Show the work that leads to your answer.

d) Sketch the graph of a function $f(x)$ that is continuous on $[1,5]$ and has an absolute minimum at $x=1$, an absolute maximum at $x=5$, a local maximum at $x=2$, and a local minimum at $x=4$. Answers will vary.

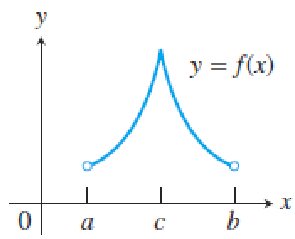
2

Determine the absolute extrema of each function.

a)



b)



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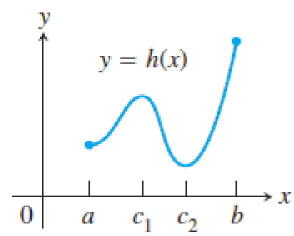


3

Find the critical values of the functions over their domain.

a) $x(t)=3t^4+4t^3-6t^2$

b) $g(t)=5t^{\frac{2}{3}}+t^{\frac{5}{3}}$

c) $f(x)=x\ln x$

d) $g(\theta)=4\theta-\tan\theta$

e) $f(x)=xe^{2x}$

4

Find the absolute extrema of f on the given interval.

a) $f(x)=2x^3-3x^2-12x+1$ on $[-2,3]$

b) $f(x)=(x^2-1)^3$ on $[-1,2]$

c) $f(x)=\sin\left(x+\frac{\pi}{4}\right)$ on $\left[0,\frac{7\pi}{4}\right]$

d) $k(x)=e^{-x^2}$ on $-\infty<x<\infty$

5

Determine the absolute maximum value of $f(x)=\dfrac{5+2x}{x^2+14}$ on the interval $[-2,4]$

6

Let *f* be the function defined by $f(x)=\sin x-\cos^2 x$ on $[0,2\pi]$.

a) Find $f'(x)$.

b) Find all critical values of $f(x)$ on $(0,2\pi)$.

c) Determine the absolute maximum value of $f(x)$ on $[0,2\pi]$.

AP Calculus BC – Worksheet 32 **Mean Value Theorem**

1 Use the function $f(x) = -x^2 + 3x + 10$ to answer the following.

a) On the interval $[2,6]$, what is the average rate of change of $f(x)$?

b) On the interval $(2,6)$, when does the instantaneous rate of change of $f(x)$ equal the average rate of change found in (a).

2 For each function, determine if the Mean Value Theorem can be applied. If so, find the value of *c* that satisfies the Mean Value Theorem.

a) $f(x) = x^2 - 5x + 2$ on $[-4,-2]$

b) $y = \sin 3x$ on $[0,\pi]$

c) $y = (-5x+15)^{1/2}$ on $[1,3]$

3

x	0	2	5	7	11
$f(x)$	13	5	17	28	41

Selected values of a differentiable function f are given in the table above. What is the fewest possible number of values of *c* in the interval $[0,11]$ for which the Mean Value Theorem guarantees that $f'(c) = 4$?

4

x	0	1	2	3
$f(x)$	0	4	7	6

Let *f* be a function with selected values given in the table above. Which of the following statements must be true?

I. By the Intermediate Value Theorem, there is a value *c* in the interval $(0,3)$ such that $f(c) = 2$.

II. By the Mean Value Theorem, there is a value *c* in the interval $(0,3)$ such that $f'(c) = 2$.

III. By the Extreme Value Theorem, there is a value *c* in the interval $[0,3]$ such that $f(c) \le f(x)$ for all *x* in the interval $[0,3]$.

5

x	$f(x)$	$f'(x)$	$g(x)$	$g'(x)$
1	3	8	2	4
2	6	3	1	2
3	5	-3	6	3
4	-2	6	3	5

The functions *f* and *g* are differentiable for all real numbers. The table above gives values of the functions and their first derivatives at selected values of *x*. The function *h* is given by $h(x) = f(g(x)) + 2$.

a) Must there be a value *r* for $2 < r < 4$ such that $h(r) = 6$?

b) Must there be a value *c* for $2 < c < 4$ such that $h'(c) = 1$?

6 A particle moves along the *x*-axis so that its position at any time $t \ge 0$ is given by $x(t) = t^3 - 3t^2 + t + 1$. For what values of *t*, $0 \le t \le 2$, is the particle's instantaneous velocity the same as its average velocity on the closed interval $[0,2]$?

AP Calculus BC – Worksheet 33 **Increasing, Decreasing, and the 1st Derivative Test**

1. For each function,
   - find the intervals where the curve is increasing and decreasing. Justify your answer.
   - Locate the x-value of each extrema and classify each as either a relative max or relative min. Justify your answer.

   a) $f(x) = x^3 - 12x + 1$

   b) $g(x) = xe^{5x}$

   c) $f(x) = (x-5)^{2/3}$

2. Let f be the function given by $f(x) = 4 - x$. g is a function with derivative given by $g'(x) = f(x)f'(x)(x-2)$. On what intervals is g decreasing?

3. Given $f'(x)$, the derivative of f, determine all x-values of relative max/min of f. No justification necessary.

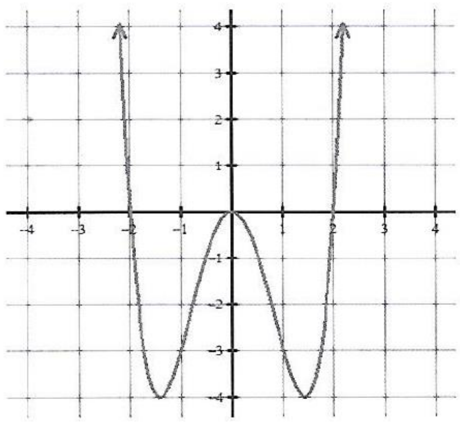
   a) $f'(x) = x^3 - 6\cos(x^2) + 2$

   b) $f'(x) = \dfrac{2 - \ln x}{x^2}$

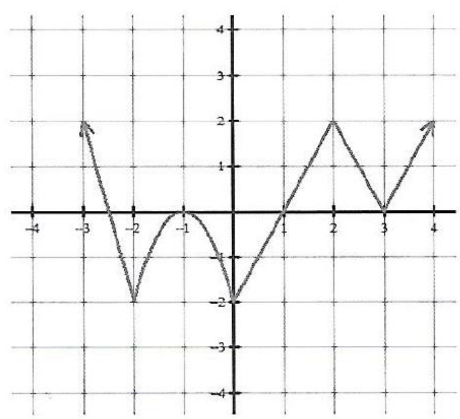
   c) $f'(x) = \sqrt{x^4 + 2} + x^2 - 5x$

4. Given the graph of $f'(x)$, the derivative of f, determine all x-values of relative max/min of f. Justify your answers.

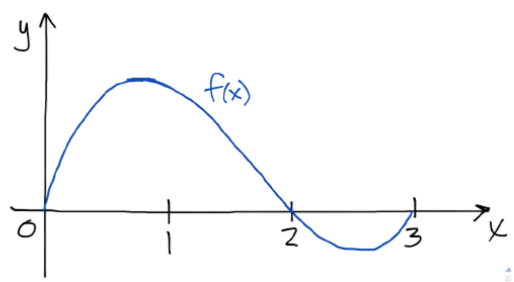
a)



b)



AP Calculus BC – Worksheet 34 **Concavity, Points of Inflection, and the 2nd Derivative Test**

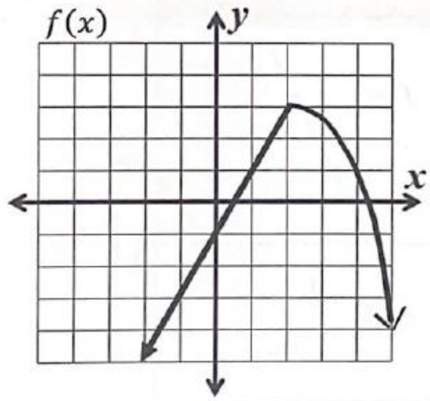
1	For $f(x)=2x^4-8x+3$, determine the value(s) of x at which $f(x)$ has a point of inflection.
2	For $f(x)=\left(2x^2+5\right)^2$, determine the value(s) of x at which $f(x)$ has a point of inflection.
3	For $f(x)=e^{-x^2}$, determine the value(s) of x at which $f(x)$ has a point of inflection.
4	For $f(x)=\sin\frac{x}{2}$, determine the value(s) of x in the interval $(-\pi,3\pi)$ at which $f(x)$ has a point of inflection.
5	Let f be a function with a second derivative given by $f''(x)=x^2(x-4)(x-7)$. What are the x-coordinates of the points of inflection of the graph of f?
6	Determine the intervals of concavity for $g(x)=\frac{x}{x-1}$?
7	For what values of x does $f(x)=x^5+5x^4+11$ have a point of inflection?
8	Let f be the function defined by $f(x)=2x^3-3x^2-12x+18$. On which intervals is the graph of f both increasing and concave down?
9	A cubic polynomial function f is defined by $f(x)=4x^3+ax^2+bx+k$, where a, b, and k are constants. The function f has a local minimum at $x=-1$, and the graph of f has a point of inflection at $x=-2$. Find the values of a and b.
10	 The graph of a differentiable function f is shown in the figure above and has an inflection point at $x=\frac{3}{2}$. Place the following values in increasing order: $f(2)$, $f'(2)$, $f''(2)$

11	For each of the following: (i) Identify the open intervals of concavity (ii) Locate the inflection points. Justify all answers. (a) $f(x)=3x^4-4x^3-6x^2+12x+1$ (b) $f(x)=\arcsin x$ on $[-1,1]$ (c) $f(x)=\cos^2 x-2\sin x$ on $[0,2\pi]$
12	Use the Second Derivative Test (if possible) to locate and **<u>justify</u>** the local extrema of the following functions: (a) $f(x)=x^3-3x^2$ (b) $f(x)=2x^2\ln x$ (c) $f(x)=e^{-x}(x-7)$ (d) $f(x)=x+2\sin x$ on the interval $(0,2\pi)$
13	Consider the differential equation $\frac{dy}{dx}=4x+y$. Find $\frac{d^2y}{dx^2}$. Determine the concavity of all solution curves for the given differential equation in Quadrant I. Give a reason for your answer.
14	Write an equation of the line tangent to the graph of $y=x^3-3x^2-4$ at its point of inflection.
15	Consider the differential equation $\frac{dy}{dx}=6-2y$. Let $y=f(x)$ be the particular solution to the differential equation with the initial condition $f(0)=4$. a) Write an equation for the line tangent to the graph of $y=f(x)$ at $x=0$. Use the tangent line to approximate $f(0.6)$. b) Is the approximation found in part (a) an overestimate or underestimate of the actual value of $f(0.6)$? Give a reason for your answer.

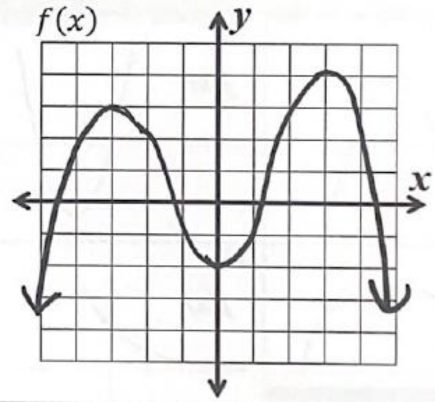
AP Calculus BC – Worksheet 35 **Analyses of graphs using f' and f''**

The graph of a function f is shown. On the same coordinate plane, sketch a graph of f', the derivative of f.

1.

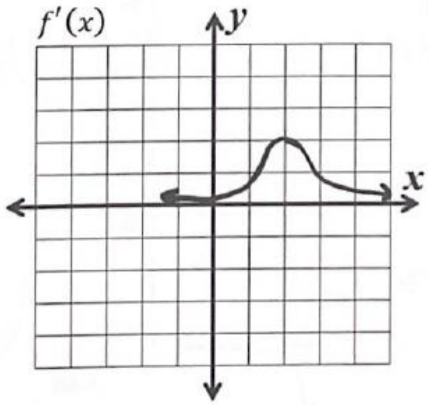


2.

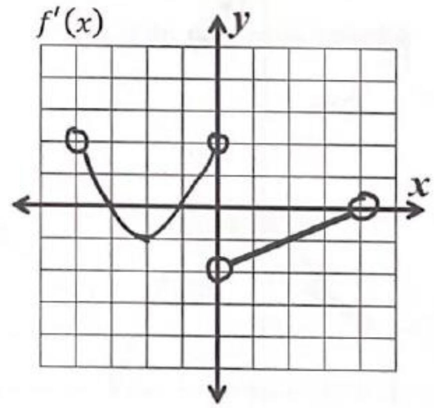


The graph of f', the derivative of f, is shown. On the same coordinate plane, sketch a possible graph of f.

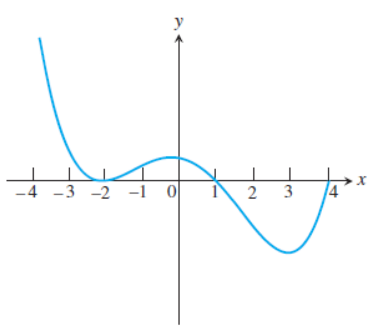
3.



4.



5. The graph of the <u>derivative</u> of f is shown below.

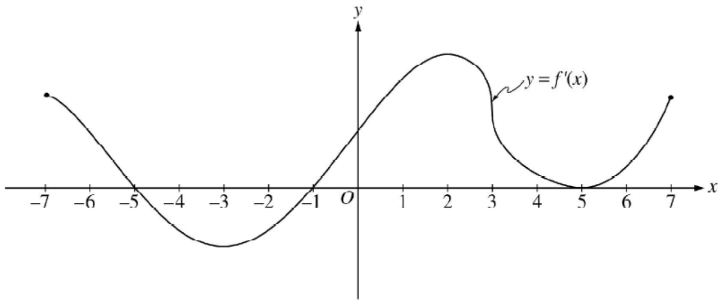


a) On what intervals is f increasing? Justify.

b) On what intervals is the graph of f decreasing? Justify.

c) At which x-coordinates does f have local extrema? Justify.

d) Sketch a possible graph of f on the interval $(-\infty, \infty)$.



6. The figure above shows the graph of f', the derivative of the function *f*, for $-7 \le x \le 7$. f' has zeroes at $x = -5, -1,$ and 5 and horizontal tangent lines at $x = -3, 2,$ and 5.

(a) Find all values of *x*, for $-7 < x < 7$, at which *f* attains a relative (local) minimum. Justify your answer.

(b) Find all values of *x*, for $-7 < x < 7$, at which *f* attains a relative (local) maximum. Justify your answer.

(c) Describe the increasing/decreasing behavior of the graph of *f*.

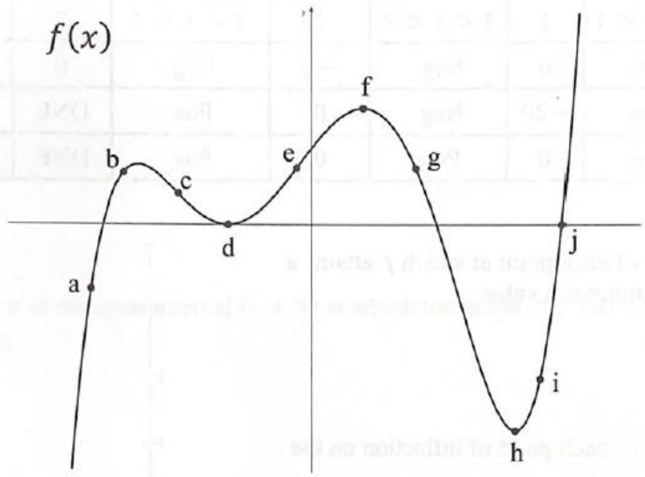
(d) Find all values of *x*, for $-7 < x < 7$, at which *f* attains a point of inflection. Justify your answer.

(e) Describe the concavity of the graph of *f*.

7. Let *f* be the function defined by $f(x) = \frac{1}{3}x^3 - 3x^2 - 16x$. On which intervals is the graph of *f* both decreasing and concave down?

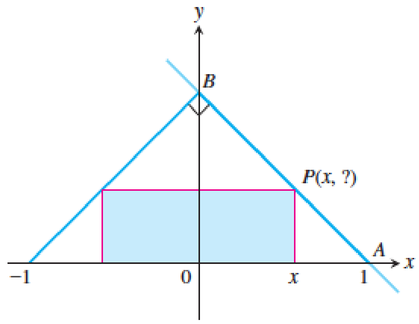
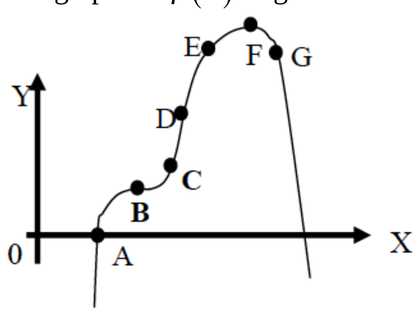
8.

Using the figure below, complete the chart by indicating whether each value is positive (+), negative (-), or zero (0) at the indicated points. For these problems, if the point appears to be a max or min, assume it is. If it appears to be a point of inflection, assume it is.

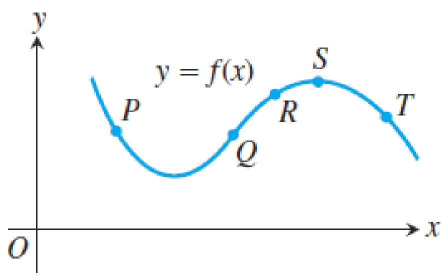


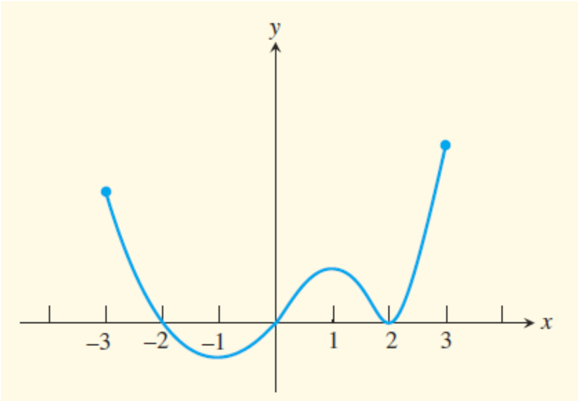
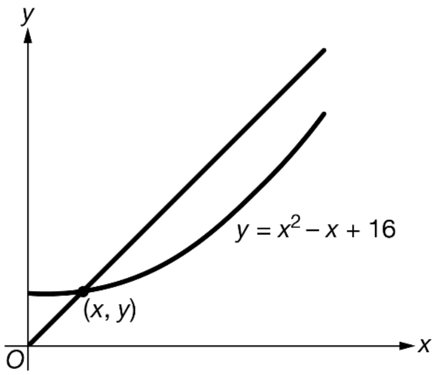
x	a	b	c	d	e	f	g	h	i	j
$f(x)$										
$f'(x)$										
$f''(x)$										

AP Calculus BC - Worksheet 36 **Optimization**

1	The figure shows a rectangle inscribed in an isosceles right triangle whose hypotenuse is 2 units long.  (a) Express the *y*-coordinate of *P* in terms of *x*. [Hint: Write an equation for the line *AB*] (b) Express the area of the rectangle in terms of *x*. (c) What is the largest area the rectangle can have, and what are the dimensions?
2	A rectangular plot of farmland will be bounded on one side by a river and on the other three sides by a single-strand electric fence. With 800 m of wire at your disposal, what is the largest area you can enclose, and what are its dimensions?
3	The height of an object moving vertically is given by $s = -16t^2 + 96t + 112$, with *s* in ft. and *t* in seconds. Find (a) the object's velocity when $t = 0$, (b) its maximum height and when it occurs, and (c) its velocity when $s = 0$.
4	You are designing a 1000-cm^3 right circular cylindrical can whose manufacturer will take waste into account. There is no waste in cutting the aluminum for the side, but the top and bottom of radius *r* will be cut from squares that measure 2*r* units on a side. The total amount of aluminum used up by the can will therefore be $A = 8r^2 + 2\pi rh$. What is the ratio of *h* to *r* that will create the most economical can?
5	What is the smallest perimeter possible for a rectangle whose area is 16 in^2, and what are its dimensions?
6	You are planning to make an open rectangular box from an 8- by 15-in. piece of cardboard by cutting congruent squares from the corners and folding up the sides. What are the dimensions of the box of largest volume you can make, and what is its volume?
	If the graph of $f(x)$ is given below for #7 - #10, give the letter for the point where:  7. $f'(x) = 0$ and $f''(x) < 0$ 8. $f'(x) > 0$ and $f''(x) = 0$ 9. $f'(x) < 0$ and $f''(x) < 0$ 10. $f'(x) > 0$ and $f''(x) > 0$

AP Calculus BC – Worksheet 37 **Analytical Applications of Derivative Review**

1	For $f(x)=3x^3-3x^2+5$, find the interval on which the function is (a) Increasing, (b) Decreasing, (c) Concave up, (d) Concave down. Then find any (e) Local extreme values (f) Inflection points.
2	Given $f'(x)=6(x+1)(x-2)^2$, find the x-coordinate of points at which f has a (a) Local maximum, (b) Local minimum, or (c) Point of inflection.
3	Find the linearization $L(x)$ of $f(x)=e^x+\sin x$ centered at $a=0$.
4	Given the graph of $f(x)$, at which of the five points shown are y' and y'' both negative? 
5	A particle is moving along a line with position function $s(t)=3+4t-3t^2-t^3$. Find the (a) velocity and (b) acceleration, and (c) describe the motion of the particle for $t\geq 0$.
6	An open-top rectangular box is constructed from a 10- by 16-in. piece of cardboard by cutting squares of equal side length from the corners and folding up the sides. Find analytically the dimensions of the box of largest volume and the maximum volume.
7	The radius of a circle is changing at the rate of $-\frac{2}{\pi}\ \frac{\text{m}}{\text{sec}}$. At what rate is the circle's area changing when $r=10$ m?
8	Show that the function $y=\sin^2 x-3x$ decreases on every interval in its domain.

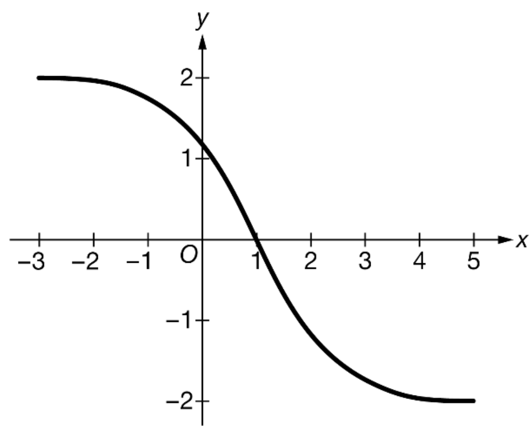
9	The accompanying figure shows the graph of the derivative of a function *f*. The doamin of *f* is the closed interval $[-3,3]$.  (a) For what values of *x* in the open interval (-3,3) does *f* have a relative maximum? Justify your answer. (b) For what values of *x* in the open interval (-3,3) does *f* have a relative minimum? Justify your answer. (c) For what values of *x* in the open interval (-3,3) is the graph of *f* concave up? Justify your answer.
10	Consider all lines in the *xy*-plane that pass through the origin and a point (x, y) on the graph of $y = x^2 - x + 16$ for $1 \le x \le 8$. The figure below shows one such line and the graph of $y = x^2 - x + 16$. Determine the equation of the line that has the smallest slope possible. 

A GRAPHING CALCULATOR IS REQUIRED FOR THIS QUESTION

1) Let *f* be a twice-differentiable function such that $f'(2)=0$. The second derivative of *f* is given by $f''(x)=x^2e^{2-x}-1$ for $0\le x\le 6$.

(a) On what open intervals contained in $0<x<6$ is the graph of *f* concave down? Give a reason for your answer.
(b) Does *f* have a relative minimum, a relative, maximum, or neither at $x=2$? Justify your answers.
(c) Use the Mean Value Theorem on the closed interval $[2,4]$ to show that $f'(4)$ cannot equal 8.5.
(d) Does the graph of *f* have a point of inflection at $x=5$? Give a reason for your answer.

NO CALCULATOR IS ALLOWED FOR THIS QUESTION



Graph of *g*

2) The graph of the continuous function *g* is shown above for $-3\le x\le 5$. The function *g* is twice differentiable except at $x=1$.

Let *f* be the function with $f(1)=3$ and derivative given by $f'(x)=(6x^2-5x)e^x$.

(a) Find the *x*-coordinate of each critical pint of *f*. Classify each critical point as the location of a relative minimum, a relative maximum, or neither. Justify your answer.
(b) Find all values of *x* at which the graph of *f* has a point of inflection. Give reasons for your answer.
(c) Fill in the missing entries in the table below to describe the behavior of g' and g'' on the interval $-3\le x\le 5$. Indicate Positive or Negative. Give reasons for your answers.

x	$x=-3$	$-3<x<1$	$x=1$	$1<x<5$	$x=5$
$g(x)$	2	Positive	0	Negative	-2
$g'(x)$	0		$-\frac{3}{2}$		0
$g''(x)$	0		Undefined		0

(d) Let *h* be the function defined by $h(x)=f(x)g(x)$. Is *h* increasing or decreasing on the interval $1<x<5$? Give a reason for your answer.

NO CALCULATOR IS ALLOWED FOR THIS QUESTION

3) The number of fish in a small bay is modeled by the function F defined by $F(t) = 10(t^3 - 12t^2 + 45t + 100)$, where t is measured in days and $0 \le t \le 8$.

(a) Using correct units, interpret the meaning of $F'(4) = -30$ in the context of the problem.

(b) Based on the model, what is the absolute minimum number of fish in the bay over the time interval $0 \le t \le 8$? Justify your answer.

(c) For what values of t, $0 \le t \le 8$, is the rate of change of the number of fish in the bay decreasing?

(d) For $0 \le t \le 8$, the number of pelicans flying near the bay is modeled by the differentiable function P, where P is a function of the number of fish in the bay. Based on the models, write an expression for the rate of change of the number of pelicans flying near the bay at the time $t = c$.